พหุนามสมมาตรในสมการกำลังสาม

Symmetric Polynomials in Cubic Equations

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บทคัดย่อ

พหุนามที่ไม่เปลี่ยนโดยการแลกเปลี่ยนหรือการสับเปลี่ยนตัวแปรในพหุนามนั้น ถูกเรียกว่า พหุนามสมมาตร ในบทความนี้ เราใช้พหุนามสมมาตรในรากของสมการกำลังสามในการหาเงื่อนไขของสัมประสิทธิ์ซึ่งทำให้สมการกำลังสาม 2 สมการมีรากร่วมกัน เราได้ว่า สมการกำลังสาม h และ H มีรากร่วมกัน ก็ต่อเมื่อ $-G^3+eG^2F-fGF^2+2fEG^2-e^2EG^2+gF^3+3gG^2-3gEFG+efEFG-3efG^2-efgEF+3efgG+2egE^2G-egEF^2+egGF-f^2GE^2+2f^2GF+fE^2F-2fgF^2-fgGE-g^2E^3+3g^2EF-3g^2G+e^3G^2-e^2fGF+e^2gF^2-2e^2gEG+ef^2EG+eg^2E^2-2eg^2F-f^3+f^2gF-fg^2E+g^3=0$

เมื่อ $h(x)=x^3+ex^2+fx+g$ และ. $H(x)=x^3+Ex^2+Fx+G$

คำสำคัญ : พหุนามสมมาตร รากร่วม สมการกำลังสาม

Abstract

A polynomial is unchanged by exchanging or interchanging the variables that occur in it, is called symmetric. In this paper, using symmetric polynomial in roots of cubic equations, we find some conditions on coefficients which two cubic equations have common roots. We obtain that the cubic equation h and H have common roots if and only if $-G^3+eG^2F-fGF^2+2fEG^2-e^2EG^2+gF^3+3gG^2-3gEFG+efEFG-3efG^2-efgEF+3efgG+2egE^2G-egEF^2+egGF-f^2GE^2+2f^2GF+fE^2F-2fgF^2-fgGE-g^2E^3+3g^2EF-3g^2G+e^3G^2-e^2fGF+e^2gF^2-2e^2gEG+ef^2EG+eg^2E^2-2eg^2F-f^3+f^2gF-fg^2E+g^3=0$

where $h(x)=x^3+ex^2+fx+g$ and. $H(x)=x^3+Ex^2+Fx+G$

Keywords : Symmetric Polynomials, Common Root, Cubic Equation

Introduction

Consider a cubic equatio n

$$ax^3+bx^2+cx+d=0$$

where a, b, c and d are coefficients and $a\neq 0$.

We can change this equation to monic polynomial equation, because we divide our cubic equations by their leading coefficients. We have

$$x^3+ex^2+fx+g=0$$

where e, f and g are coefficients.

Let r_1, r_2 and r_3 be roots of the cubic equation. We have

$$x^3+ex^2+fx+g=(x-r_1)(x-r_2)(x-r_3)$$

By multiplying and grouping terms gives.

$$x^3+ex^2+fx+g=x^3-(r_1+r_2+r_3)x^2+ +(r_1r_2+r_2r_3+r_1r_3)x-r_1r_2r_3$$

By equating coefficients, we get that

$$r_1+r_2+r_3=-e,$$

$$r_1r_2+r_2r_3+r_1r_3=f$$

and $r_1r_2r_3=-g.$

We are interesting in the fact that all these polynomials are *symmetric*. A symmetric polynomial is unchanged by exchanging or interchanging the variables that occur in it [1].

In 2003, George G. St. find some condition on coefficients which two quadratic equations have common roots by using symmetric polynomials of roots see [2]. He obtain that the quadratic equations f and g have common roots if and only if

$$e^2-deD+d^2E-2eE+eD^2 -dDE+E^2=0$$

where $f(x)=x^2+dx+e=0$ and

$$g(x)=x^2+Dx+E=0.$$

In this work, we use symmetric polynomials of roots to find some conditions on coefficients which two cubic equations have common roots.

Results

In this section, we present some condition on coefficients which two cubic equations have common roots. We begin with the following theorem.

<u>Main Theorem</u>:

Let h and H be cubic equations, write $h(x)=x^3+ex^2+fx+g$ and $H(x)=x^3+Ex^2+Fx+G$ where e, f, g, E, F and G are coefficients. Then h and H have common roots if and only if

$$-G^3+eG^2F-fGF^2+2fEG^2$$
$$-e^2EG^2+gF^3+3gG^2-3gEFG$$
$$+efEFG-3efG^2-efgEF$$
$$+3efgG+2egE^2G-egEF^2$$
$$+egGF-f^2GE^2+2f^2GF$$
$$+fgE^2F-2fgF^2-fgGE$$
$$-g^2E^3+3g^2EF-3g^2G+e^3G^2$$
$$-e^2fGF+e^2gF^2-2e^2gEG$$
$$+ef^2EG+eg^2E^2-2eg^2F-f^3G$$
$$+f^2gF-fg^2E+g^3=0.$$

<u>Proof</u>: Assume that r_1, r_2 and r_3 are roots of H. We have

$$H(x)=x^3+Ex^2+Fx+G$$
$$=(x-r_1)(x-r_2)(x-r_3)$$

By calculating, we obtain

$$x^3+Ex^2+Fx+G$$
$$=x^3-(r_1+r_2+r_3)x^2$$
$$+(r_1r_2+r_2r_3+r_1r_3)x$$
$$-r_1r_2r_3.$$

By equating coefficients, we have

$$r_1+r_2+r_3=-E,$$
$$r_1r_2+r_2r_3+r_1r_3=F$$

and $r_1r_2r_3=-G.$

We can be written in terms of the coefficients of H:

$$F^2-2EG$$
$$=r_1^2r_2^2+r_2^2r_3^2+r_1^2r_3^2,$$
$$F^3+3G^2-3EFG$$
$$=r_1^3r_2^3+r_2^3r_3^3+r_1^3r_3^3,$$
$$-EF+3G$$

$=r_1^2r_2+r_1r_2^2+r_2^2r_3$

$+ r_2r_3^2+r_1^2r_3+r_1r_3^2,$

$2GE^2-EF^2+GF$

$=r_1^3r_2^2+r_1^2r_2^3+r_2^3r_3^2$

$+ r_2^2r_3^3+r_1^3r_3^2+r_1^2r_3^3,$

E^2-2F

$$= r_1^2 + r_2^2+r_3^2,$$

E^2F-2F^2-GE

$=r_1^3r_2+r_1r_2^3+r_2^3r_3$

$+ r_2r_3^3+r_1^3r_3+r_1r_3^3,$

and

$-E^3+3EF-3G$

$$= r_1^3+r_2^3+ r_3^3. \qquad (*)$$

Suppose that r_1, r_2 or r_3 are common roots of h. We have

$$h(r_1)\cdot h(r_2)\cdot h(r_3)=0.$$

Then $(r_1^3+er_1^2+f r_1+g)$

$\cdot(r_2^3+er_2^2+f r_2+g)$

$\cdot(r_3^3+er_3^2+f r_3+g)$

$=0.$

By multiplying, grouping terms and substituting with $(*)$ becomes

$-G^3+eG^2F-fGF^2+2fEG^2$

$-e^2EG^2+gF^3+3gG^2-3gEFG$

$+efEFG-3efG^2-efgEF$

$+3efgG+2egE^2G-egEF^2$

$+egGF-f^2GE^2+2f^2GF$

$+fgE^2F-2fgF^2-fgGE$

$-g^2E^3+3g^2EF-3g^2G+e^3G^2$

$-e^2fGF+e^2gF^2-2e^2gEG$

$+ef^2EG+eg^2E^2-2eg^2F-f^3G$

$+f^2gF-fg^2E+g^3=0. \quad (**)$

Hence, we obtain that the statement in this theorem is true.

<u>Example:</u>

Consider the cubic equations

$2x^3-3x^2-3x+2$

$=(2x-1)(x+1)(x-2)$

and

x^3-3x^2-x+3

$=(x-3)(x+1)(x-1).$

We see that they share the root $x=-1$. To test by use the Main Theorem, we divide by the lead coefficients. We get

$2x^3-3x^2-3x+2=0$ to change

$x^3-\frac{3}{2}x^2-\frac{3}{2}x+1=0$.

Let $H(x)=x^3-\frac{3}{2}x^2-\frac{3}{2}x+1$ and

$h(x)=x^3-3x^2-x+3$.

Then $E=-\frac{3}{2}$, $F=-\frac{3}{2}$, $G=1$,

$e=-3$, $f=-1$ and $g=3$.

Substituting E, F, G, e, f and g into formula (**). We obtain

$$-1^3+(-3)(1)^2(-\frac{3}{2})-(1)(-1)\left(-\frac{3}{2}\right)^2$$

$$+2(-1)(-\frac{3}{2})(1)^2-(-3)^2\left(-\frac{3}{2}\right)(1)^2$$

$$+3\left(-\frac{3}{2}\right)^3+3(3)(1)^2$$

$$-3(3)\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)(1)$$

$$+(-3)(-1)\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)(1)$$

$$-3(-3)(-1)(1)^2$$

$$-(-3)(-1)(3)\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)$$

$$+3(-3)(-1)(3)(1)+2(-3)(3)\left(-\frac{3}{2}\right)^2(1)$$

$$-(-3)(3)\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)^2$$

$$+(-3)(3)(1)\left(-\frac{3}{2}\right)-(-1)^2(1)\left(-\frac{3}{2}\right)^2$$

$$+2(-1)^2(1)\left(-\frac{3}{2}\right)+(-1)(3)\left(-\frac{3}{2}\right)^2\left(-\frac{3}{2}\right)$$

$$-2(-1)(3)\left(-\frac{3}{2}\right)^2-(-1)(3)(1)\left(-\frac{3}{2}\right)$$

$$-(3)^2\left(-\frac{3}{2}\right)^3+3(3)^2\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)$$

$$-3(3)^2(1)+(-3)^3(1)^2$$

$$-(-3)^2(-1)(1)\left(-\frac{3}{2}\right)+(-3)^2(3)\left(-\frac{3}{2}\right)^2$$

$$-2(-3)^2(3)\left(-\frac{3}{2}\right)(1)$$

$$+(-3)(-1)^2\left(-\frac{3}{2}\right)(1)+(-3)(3)^2\left(-\frac{3}{2}\right)^2$$

$$-2(-3)(3)^2\left(-\frac{3}{2}\right)-(-1)^3(1)$$

$$+(-1)^2(3)\left(-\frac{3}{2}\right)-1(3)^2$$

$$\left(-\frac{3}{2}\right)+(3)^3$$

=-1+9/2+9/4+3+27/2-81/8+9-81/4+27/4

$$-9-\frac{81}{4}+27-\frac{81}{2}-\frac{243}{8}+\frac{27}{2}-\frac{9}{4}-3$$
$$+\frac{81}{8}+\frac{27}{2}-\frac{9}{2}+\frac{243}{8}+\frac{243}{4}-27-27$$
$$-\frac{27}{2}+\frac{243}{4}+81+\frac{9}{2}-\frac{243}{4}-81+1$$
$$-\frac{9}{2}-\frac{27}{2}+27$$
$$=0$$

Thus, we have that two cubic polynomials equations have a common root.

We see that we determine two cubic polynomials have common root by use their coefficients, without having to find the roots of the polynomials. In other polynomials, we generalize idea of symmetric polynomial of roots to find condition on coefficients which two polynomials have common roots.

Reference

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